Handout: Maxwell's Equations in 3D

In the main notes we have examined Maxwell's Equations for electromagnetism in restricted dimension (The electric field E is along x and the magnetic field B is along$y$. Both vary only in time and along one direction z. These equations can be naturally extended into a general three-dimensional space. However, this is most often done with a set of differential vector operators, called *grad* (from *gradient*), *div* (from *divergence*) and *curl*:

3.1 Grad: ∇

$$\nabla = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \tag{3.1}$$

We can apply the ∇ operator to other vectors (e.g. $V(x, y, z)$) using either the dot product or cross product:

3.2 Div: $\nabla \cdot V$

$$\nabla \cdot V = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \cdot \begin{pmatrix} V_x(x,y,z) \\ V_y(x,y,z) \\ V_z(x,y,z) \end{pmatrix} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \tag{3.2}$$

3.3 Curl: $\nabla \times V$

$$\nabla \times V = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \times \begin{pmatrix} V_x(x,y,z) \\ V_y(x,y,z) \\ V_z(x,y,z) \end{pmatrix} = \begin{pmatrix} \frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \\ \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \\ \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \end{pmatrix} \tag{3.3}$$

With these operators defined, we can write down Maxwell's Equations in differential form (you may also see these in terms of $D = \epsilon E$ and $H = B/\mu$):

Maxwell's Equations	
1	$\nabla \cdot \mathbf{E} = \rho/\epsilon$
2	$\nabla \cdot \mathbf{B} = 0$
3	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
4	$\nabla \times \mathbf{B} = \mu \mathbf{J} + \mu\epsilon \frac{\partial \mathbf{E}}{\partial t}$

3.4 Relation to simplified equations in main notes

First we should convince ourselves that in the restricted dimensions described in the notes, we get the simplified equations we have used. In the notes we assumed that E and B vary only in time and along one direction z, so $\partial/\partial x = 0$ and $\partial/\partial y = 0$. This gives us the following version of **Maxwell 3**:

$$\nabla \times E = \begin{pmatrix} -\frac{\partial E_y}{\partial z} \\ \frac{\partial E_x}{\partial z} \\ 0 \end{pmatrix} = -\begin{pmatrix} \frac{\partial B_x}{\partial t} \\ \frac{\partial B_y}{\partial t} \\ 0 \end{pmatrix} \tag{3.4}$$

(And a similar expression for **Maxwell 4**). We can then just pick one pair of elements in the above relation, and see that for an E field along x, say, we have the relation:

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t}, \tag{3.5}$$

as we have used in the notes.

3.5 Electromagnetic waves

Now let's show that the general Maxwell's equations produce electromagnetic waves in the absence of any charges or currents. We'll need to use a vector identity:

$$\nabla \times \nabla \times V \equiv \nabla(\nabla \cdot V) - \nabla \cdot \nabla \cdot V, \tag{3.6}$$

where the double differential $\nabla \cdot \nabla$ is often written as:

$$\nabla \cdot \nabla = \nabla^2 = \begin{pmatrix} \partial^2/\partial x^2 \\ \partial^2/\partial y^2 \\ \partial^2/\partial z^2 \end{pmatrix} \tag{3.7}$$

From **Maxwell 3**, we have :

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{3.8}$$

Take curl of both sides:

$$\nabla \times \nabla \times \mathbf{E} = -\nabla \times \left(\frac{\partial \mathbf{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) \tag{3.9}$$

We can simplify the left hand side of the above using **Maxwell 1** with the charge density $\rho = 0$:

$$\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\nabla^2 \mathbf{E} \tag{3.10}$$

And **Maxwell 4** (with the current $\mathbf{J} = 0$) tells us

$$-\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = -\frac{\partial}{\partial t}\left(\mu\epsilon\frac{\partial \mathbf{E}}{\partial t}\right) = -\mu\epsilon\frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{3.11}$$

Thus Maxwell's equations satisfy the general form the wave equation:

$$\nabla^2 \mathbf{E} = \mu\epsilon\frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{3.12}$$

With the velocity:

$$\frac{1}{v^2} = \mu\epsilon, \quad \text{so} \quad v = \frac{1}{\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{\mu_0\mu_r\epsilon_0\epsilon_r}} \tag{3.13}$$